

# Street-Fighting Mathematics

The Art of Educated Guessing and  
Opportunistic Problem Solving

*Sanjoy Mahajan*

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